Lecture 8

13.2 - Integrals and Derivatives

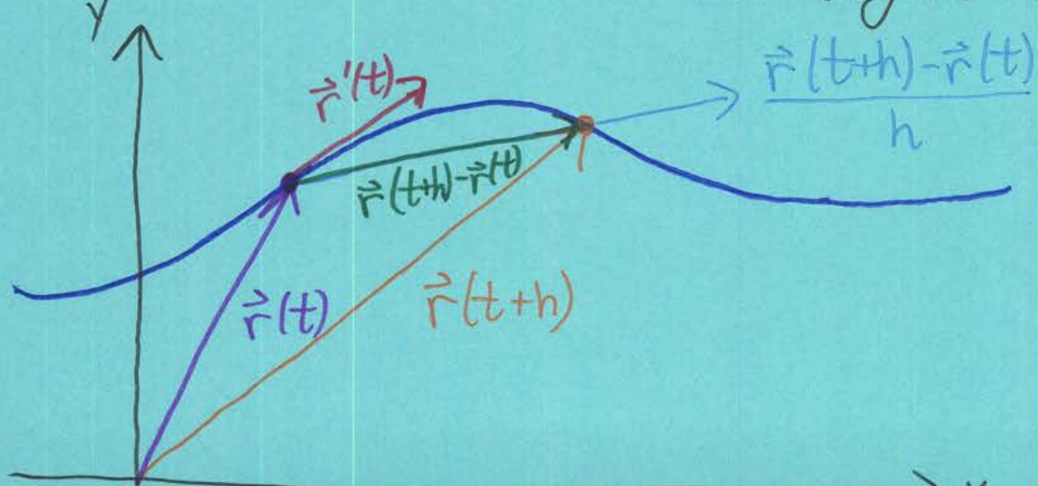
In Calc I, after limits, you learned about differentiation.

<u>Def</u>: The <u>derivative</u> of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is:

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

Just as the derivative gave you the slope of the tangent line at a point in calc I, here it gives the direction vector of the tangent line.



So, $\vec{r}\,'(a)$ is the tangent vector to $\vec{r}(t)$ at $t=a$, (provided $\vec{r}\,'(a) \neq \vec{0}$). It will also be useful to consider the unit tangent vector:

$$\vec{T}(t) := \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

We will spend a lot of time with this in the next section.

Ex: Let $\vec{r}(t) = \langle t\cos t,\ t,\ t\sin t \rangle$.

i) Find $\vec{r}\,'(t)$.

ii) Find $\vec{T}(t)$.

iii) Find an equation for the tangent line to $\vec{r}(t)$ at $t=\pi$.

Sol:

i) $\vec{r}\,'(t) = \langle \cos t - t\sin t,\ 1,\ \sin t + t\cos t \rangle$

ii) $|\vec{r}\,'(t)| = \sqrt{(\cos t - t\sin t)^2 + 1^2 + (\sin t + t\cos t)^2}$

$$= \sqrt{\cos^2 t \cancel{- 2t\cos t\sin t} + t^2\sin^2 t + 1 + \sin^2 t \cancel{+ 2t\sin t\cos t} + t^2\cos^2 t}$$

$$= \sqrt{1 + t^2 + 1} = \sqrt{t^2+2}$$

So

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{t^2+2}}\langle \cos t - t\sin t,\ 1,\ \sin t + t\cos t\rangle$$

iii) We need a tangent vector at $t=\pi$. Notice we could use either of $\vec{r}\,'(\pi)$ or $\vec{T}(\pi)$, but since $\vec{r}\,'(t)$ is simpler, let's find that one.

$$\vec{r}\,'(\pi) = \langle \cos\pi - \pi\sin\pi,\ 1,\ \sin\pi + \pi\cos\pi\rangle = \langle -1, 1, -\pi\rangle$$

Now, $\vec{r}(t)$ passes through

$$\vec{r}(\pi) = \langle \pi\cos\pi,\ \pi,\ \pi\sin\pi\rangle = \langle -\pi, \pi, 0\rangle$$

So, the tangent line is

$$\vec{\ell}(s) = \underbrace{\vec{r}(\pi)}_{\text{initial point}} + s\underbrace{\vec{r}\,'(\pi)}_{\text{direction vector}} = \langle -\pi, \pi, 0\rangle + s\langle -1, 1, \pi\rangle = \langle -\pi - s, \pi + s, \pi s\rangle$$

(We use s here as the parameter since $\vec{r}$ is already using t.)

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Though I don't recommend memorizing this as a formula, but rather understanding why this is the formula, the equation for the tangent line to $\vec{r}(t)$ at $t=a$ is

$$\vec{\ell}(s) = \vec{r}(a) + s\,\vec{r}\,'(a).$$

1) $\dfrac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = \vec{u}\,'(t)+\vec{v}\,'(t)$

2) $\dfrac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\vec{u}\,'(t)$

3) $\dfrac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}\,'(t)$

4) $\dfrac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$

5) $\dfrac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$

Product rule

6) $\dfrac{d}{dt}\left[\vec{u}(f(t))\right] = \vec{u}\,'(f(t))\,f'(t).$ Chain rule

Let's see # 4 in action: Assume $\vec{r}\,'(t) \neq \vec{0}$.

$$\frac{d}{dt}\left[|\vec{r}(t)|\right] = \frac{d}{dt}\left[\sqrt{\vec{r}(t)\cdot\vec{r}(t)}\right] = \frac{1}{2}\,\frac{\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right]}{\sqrt{\vec{r}(t)\cdot\vec{r}(t)}}$$

$$= \frac{1}{2}\,\frac{2\vec{r}(t)\cdot\vec{r}\,'(t)}{|\vec{r}(t)|} = \frac{\vec{r}(t)\cdot\vec{r}\,'(t)}{|\vec{r}(t)|}$$

Just as we define integrals in calc I, we can do so here too:

<u>Def</u>: The <u>definite integral</u> of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is

$$\int_a^b \vec{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{r}(t_i^*)\Delta t$$

$$= \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle.$$

We can also define <u>indefinite integrals</u> as usual.

<u>Ex</u>: Find $\int \vec{r}(t)\,dt$ and $\int_0^2 \vec{r}(t)\,dt$ where $\vec{r}(t) = t\,\hat{i} - t^3\hat{j} + 3t^5\hat{k}$.

<u>Sol</u>: $\int \vec{r}(t)\,dt = \left(\int t\,dt\right)\hat{i} - \left(\int t^3\,dt\right)\hat{j} + \left(\int 3t^5\,dt\right)\hat{k}$

$$= \left(\tfrac{1}{2}t^2 + c_i\right)\hat{i} - \left(\tfrac{1}{4}t^3 + c_j\right)\hat{j} + \left(\tfrac{1}{2}t^6 + c_k\right)\hat{k}$$

If we write $\vec{C} = \langle c_i, c_j, c_k \rangle$, then we can also write

$$\int \vec{r}(t)\,dt = \tfrac{1}{2}t^2\hat{i} - \tfrac{1}{4}t^3\hat{j} + \tfrac{1}{2}t^6\hat{k} + \vec{C}.$$

$$\int_0^2 \vec{r}(t)\,dt = \left(\tfrac{1}{2}t^2\Big|_0^2\right)\hat{i} - \left(\tfrac{1}{4}t^3\Big|_0^2\right)\hat{j} + \left(\tfrac{1}{2}t^6\Big|_0^2\right)\hat{k}$$

$$= 2\hat{i} - 2\hat{j} + 32\hat{k}$$